Exercise Session 5

(1) $f: X \to S$, $\mathcal{L}$ line bundle on X, $t_0, \dots, t_n \in \Gamma(X, \mathcal{L})$ generating $\mathcal{L}$ (i.e. $\mathcal{O}_X^{n+1} \to \mathcal{L}$ surj.)

(a) Construct $t: X \to \mathbb{P}^n_S$ from $t_0, \dots, t_n$

- Wlog $S = \operatorname{Spec} A$ is affine (in general case glue over cover of S)
- Let

$$X_i := D(t_i) = \{x \in X \mid (t_i)_x \notin m_x \mathcal{L}_x\} \subseteq X \qquad \text{for } i = 0, \dots, n$$

Then $X_i \subset X$ open and $X = \bigcup_i X_i$.

- Then let

$$f_i : X_i \longrightarrow \underset{\substack{\cong \\ \operatorname{Spec} A\left[\frac{T_0}{T_i}, \dots, \frac{T_n}{T_i}\right]}}{D_+(T_i)} \subseteq \mathbb{P}^n_A \quad , \qquad \frac{T_j}{T_i} \mapsto \frac{t_j}{t_i} \in \Gamma(X_i, \mathcal{O}_{X_i})$$

- Obviously glues.

(b) Assume $f: X \to S$ is proper and all $t(s): X_s \to \mathbb{P}^n_{k(s)}$ are closed immersions. Then t is a closed immersion.

$$\begin{tikzcd} X_z \arrow[r] \arrow[d, hook] & X_s \arrow[r] \arrow[d, hook, "t(s)"'] & X \arrow[d, "t"] \\ z \arrow[r] & \mathbb{P}^n_{k(s)} \arrow[r] \arrow[d] & \mathbb{P}^n_S \arrow[d] \\ & s \arrow[r] & S \end{tikzcd}$$

From the assumption on $t(s)$'s, t has finite fibers (in fact t is injective on top. spaces).

Have

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[dr, "\text{proper}"'] & \mathbb{P}^4_S \arrow[d, "\text{proper}"] \\ & S \end{tikzcd} \Rightarrow t \text{ proper (by valuative criterion for properness)}$$

Stacks, Lemma 02LS $\Longrightarrow$ t is finite

$\forall z \in \mathbb{P}^4_S$ over some $s \in S$, have $X_s \hookrightarrow \mathbb{P}^4_{\kappa(s)}$ is closed immersion.

apply to $t: X \to \mathbb{P}^4_S$: $\rightsquigarrow X_z \to z$ closed immersion.

<u>Lemma</u>: Let $\varphi: Y \to Z$ be a finite map of schemes s.t. $\forall z \in Z$, $Y_z \to z$ is a closed immersion. Then φ is a closed immersion.

<u>Proof</u>: Wlog $Z = \operatorname{Spec} C$ affine. Then φ finite $\Rightarrow Y = \operatorname{Spec} B$ affine and B finite as a C-module. Need to show: $C \to B$ surj.

Can be checked on stalks, i.e. need to see $C_p \to B_p$ surj. $p \in Z$.

By Nakayama, follows from $C_p/p \to B_p/pB_p$ surj. (by assumption)

$\hat{=}\ Y_z \to z$ for $z = p$. □

(2) Let S noeth. scheme, $f: X \to S$ relative curve of genus 1. Let $e: S \to X$ be a section of f. Then locally on S,

$$X \cong V_+(y^2 + a_1 xy + a_3 y - x^3 - a_2 x^2 - a_4 x - a_6) \subset \mathbb{P}^2_S$$

for some $a_i \in \Gamma(S, \mathcal{O}_S)$.

Can assume $S = \operatorname{Spec} A$ affine. Look at $\mathcal{L}_i = \mathcal{O}_X(i \cdot e) = \mathcal{O}_X(\Gamma_e)^{\otimes i}$.

Then $\Gamma(X, \mathcal{L}_i)$ is a fin. proj. A-module by "Base-Change lecture"

$\rightsquigarrow$ after shrinking S, can assume that $\Gamma(X, \mathcal{L}_i)$ is free $/A$.

By RR on fibers,

$$\mathrm{rk}_A\, \Gamma(X, \mathcal{O}_X(i \cdot e)) = \begin{cases} 1 & i=0 \\ i & i \geq 1 \end{cases}$$

Pick

$1 \in \Gamma(X, \mathcal{O}_X)$

$x \in \Gamma(X, \mathcal{O}_X(2e)) \setminus \Gamma(X, \mathcal{O}_X)$

$y \in \Gamma(X, \mathcal{O}_X(3e)) \setminus \Gamma(X, \mathcal{O}_X(2e))$

By shrinking S again, can assume that $1, x, y$ is basis of $\Gamma(X, \mathcal{O}_X(3e))$.

$\rightsquigarrow 1, x, y, x^2, xy, x^3 \in \Gamma(X, \mathcal{O}_X(6e))$ is basis (since it's true in every fiber)

$\rightsquigarrow 1, \dots, xy, y^2 \in \Gamma(X, \mathcal{O}_X(6e))$ is basis

$\Rightarrow x^3 - \lambda y^2 \in \mathrm{span}_A(1, x, y, x^2, xy)$ for some $\lambda \in A^\times$

Replace $x \mapsto \lambda^{-1} x$, $y \mapsto \lambda^{-1} y$ to get

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

for some $a_i \in A$.

By 1, $1, x, y \in \mathcal{O}_X(3e)$ induce a closed immersion

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \mathbb{P}^2_S \arrow[d] \\ & S \end{tikzcd}$$

By looking at fibers, have $X = V_+(\dots)$.

③ (a) k field, C/k proper smooth connected curve of genus 0.
Assume $\exists P \in C(k)$. Then $C \cong \mathbb{P}^1_k$.

By Riemann-Roch,

$$\underbrace{\chi(\mathcal{L})}_{= h^0(\mathcal{L}) - \underbrace{h^1(\mathcal{L})}_{\overset{\text{Serre duality}}{=} h^0(\mathcal{L}^\vee \otimes \Omega^1) = 0 \text{ if } \deg \mathcal{L} \geq -1}} = \deg \mathcal{L} + 1$$

(then $\deg(\mathcal{L}^\vee \otimes \Omega^1) < 0$)

$\rightsquigarrow$ If $\deg \mathcal{L} \geq -1$, then

$$h^0(\mathcal{L}) = \deg \mathcal{L} + 1$$

Apply this to $\mathcal{L} = \mathcal{O}_C(P)$. Then $\deg \mathcal{L} = 1$

$\rightsquigarrow$ $\exists$ non-constant $f \in \Gamma(C, \mathcal{L})$

$\rightsquigarrow$ Get $f : C \longrightarrow \mathbb{P}^1_k$.

By def. of $\mathcal{L}$, f has single pole of order 1. Thus

$$\deg f = \sum_{x \in f^{-1}(\infty)} e_x = e_P = 1$$

$\Rightarrow f$ is isom.

(b) As in ②.